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On systemic perspective and the strategies in E-learning: inquiries in linear algebra

Larissa Sbitneva[a]*, Nehemías Moreno[b], Daniel Rivera[a], Rogelio Valdez[a]

[a]*Center of Investigation in Sciences, Morelos State University, Av. Universidad 1001, Col.Chamilpa, Cuernavaca 62209, México*
[b]*Institute of Sciences in Education, Morelos State University, Av. Universidad 1001, Col.Chamilpa, Cuernavaca 62209, México*

Abstract

There is presented a teaching experience described by means of some theoretic elements of the Ontosemiotic Approach. We propose a series of problem-situations of exploration type which require an integral approach to *intra*-mathematical inquires, with investigation objectives, which involve applications of theore*tical constructions and methods* on the junctions in standard courses, in another environment. Our strategy is to design a *series of theoretical questions,* which lead to research activity, and to carry the students toward systemic perspective of mathematical practices through the practices of unitary perspective. We consider the Ontosemiotic Approach as an adequate theoretical framework which allows us to analyse the learning process in the e-modality of Linear Algebra course in order to arrange the activities so that the students would achieve the deeper comprehension of the most important concepts and their mutual relations.

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1. Introduction

The most important ideas in linear algebra course to be learned by students are related to the concepts of vector (linear) spaces and linear transformations, which presents certain difficulties due to the abstract nature of the matter.

* Sbitneva Larissa Tel.: +52-777-416-8460; +52-777-313-4466
E-mail address: larissa@uaem.mx

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Here we describe a series of activities which will allow the students to achieve the deeper comprehension of the most important concepts and their mutual relations.

1.1. Preliminary consideration of problems in the linear algebra course

Analyzing lists of problems in linear algebra courses and th**e** results in recent researches in mathematics education, we noted that there is a lack of problems which would require creative applications of the methods and technics traditionally taught in these courses. It is rather difficult to find the problems which would indicate the direction of applications of the theoretical facts beyond the themes covered in standard courses, as well as such sort of problems which would involve interdisciplinary considerations. Here we propose some tasks of exploration type which require an integral approach to intra-mathematical inquires with investigation objectives that involve applications of theoretical constructions and methods on the junctions in standard courses at the undergraduate level.

1.2. Towards intra and extra-mathematical applications within standard themes of linear algebra courses

In linear algebra courses, students should be familiarized with some operational methods related to vector (linear) spaces and linear transformations. We suggest some preliminary problem-situations of extra or intra-mathematical applications at the early stages of the course, in order to involve students in the mathematical activities of “personal cognitions” and to draw their attention to an alternative point of view on the same mathematical situation. For example, traditionally the matrices appear as a useful tool to represent linear transformations of vector spaces, nevertheless some sets of matrices may be treated in another environment, where the underlying geometry, topology and analytical methods should be involved.

We propose the implementation of the Forum scenery in order to involve all participants of e-modality course into the discursive practice of problem-situations concerned to the variety of characteristics of linear transformations corresponding to symmetric matrices and to antisymmetric ones which are most important in applications to mechanics, as well the orthogonal and unimodular matrices which describe geometric transformations used in applications (especially in the computer 3D Vision). We consider pertinent an implementation of Web2 technologies such as a WebQuest, which is considered as an inquiry-oriented lesson format, as well as Forum of general discussion of fundamental concepts.

2. Theoretical Framework

We consider the Theory of Ontosemiotic Approach as an adequate framework which allows us to analyze the learning process in the e-modality of Linear Algebra course in order to arrange the activities so that the students would achieve the deeper comprehension of the most important concepts and their mutual relations.

2.1. Theoretical background of the Ontosemiotic Approach

The Ontosemiotic Approach (OSA), considered as a unified framework to the study of the cognitive and instructional phenomena, emphasizes the role of mathematical activity which is modeled in terms of systems of practices (operative and discursive, oriented to problems solution), configurations of primary objects and processes (Godino, Font, Wilhelmi & De Castro, 2009; Font, Planas & Godino, 2010). In the process of realization of these practices there participate different types of *primary objects* (problems, language, arguments, concepts, propositions and procedures), which are organized in epistemic or cognitive configurations (depending on whether the institutional o personal level is taken into considerations), which in turn produce others mathematical objects of higher level complexity. The problem-situations promote and contextualize the mathematical activity; languages (symbols, notations and graphics) represent the other entities and serve as tools for action; arguments justify the procedures, meanwhile propositions relate the concepts. On the other hand the objects that appear in mathematical activity and those, of more complex nature, which emerge from these practices, depend on the “language game” (Wittgenstein, 1953) in which they participate and might be considered from the five facets of dualities proposed in OSA. It is important to take into account that choosing some entity into consideration as a primary one is rather relative, there is no absolute distinction (Godino, Batanero & Font, 2007), because all of them are functional entities and are related with respect to game languages (institutional frameworks and contextual peculiarity).

To analyze the mathematical activity realized by the students in the process of resolution of problems, within the OSA theory there considered not only the practices y configurations but also the processes derived from the application of the process-product duality to the configurations together with other relevant processes such that visualization (Godino, Cajaraville, Fernández & Gonzato, 2012), idealization (Font & Contreras, 2008), argumentation, algoritmization, etc.. We employ the notion of epistemic configurations in Onosemiotic Approach in order to realize an analysis of the mathematical practices performed by the students in the process of problems solving.

2.2. Description of the inquiry-oriented proposal

From the perspective adopted in this work the resolution of problem-situations is considered as the ultimate goal of the mathematical activity. Our strategy is to implement a series of theoretical problem-situations, designed with an emphasis on investigation activity, and to carry the students toward systemic perspective of mathematical practices through the practices of unitary perspective (resolution of standard typical problems traditionally employed in the linear algebra courses). In other words, with each problem-situation there is associated a practice and an epistemic configuration which permits its realization. Being rather different in essence, there is presented some articulations between them so that it is possible to establish some relations that depend on the level of the generalization. Taking into account unitary-systemic duality, the relations of articulation, by means of the process of generalization, reveal the complexity of the mathematical object under consideration (the matrices in our case) and the variety of the spaces where they play important role in the intra-mathematical reality. (Rondero & Font, 2015).

The introduction of the complexity-articulation dialectic in the learning process of a mathematical object (by means of notions of epistemic configuration and of articulation) gives an opportunity to generate criteria of the quality of mathematics in the learning process of the object. From one hand, the complexity allow to generate a criterion of the representativity (taking into consideration the sequence of tasks as a sample of representativity of the complexity of the object that is to be taught), from the other hand, the processes of articulation permit concretize the process of connection, which is one of the processes that allow to speak of the beauty of mathematics involved in some sequence of the assigned tasks.

The process of problems-solving requires the interaction of other processes: argumentations (in each of the unitary practices and between them), reification (in order to organize the passing from unitary entities to the systemic one) and algorithmization (procedures). In order to achieve a systemic practice of higher complexity through the process of reification the students have to realize the variety of unitary practices. The argumentation processes establish relations between these unitary practices by means of consideration of certain properties (formulated within the propositions). From the systemic perspective these unitary practices should be organized as the sequences in an order dictated by the algorithmic processes, thus accomplishing the systemic practice, which permits the students to achieve the comprehension of the global significance of the fundamental concepts of Linear Algebra in intra-mathematical contexts. In general, the systemic practice gives rise to the epistemic configuration which takes into account all the objects which participate in each unitary practice.

3. Epistemic configuration for the task concerned the set of invertible matrices

In this section we would like to illustrate how primary mathematical objects may be organized in epistemic configurations. For example, the object *proposition* is encountered among the links (arguments) which relate one practice with another, meanwhile each step, realized within the object *procedure*, corresponds to every one of the practices organized in the sequences.

Traditionally the epistemic configurations are presented as a table in full page format that require more printed space.

3.1. Linear transformations and the group structure in the set of invertible matrix

Problem-situation1.

Consider the set of matrices corresponding to the linear transformations group $GL(n, R)$ as a subset of $M(n, R)$ and describe the variety of mathematical constructions for R^N such as coordinates, metric, topology, neighborhood, regions, series, convergence, etc., in the new environment of $M(n, R)$ and $GL(n, R)$.

Language: vector space, basis, Euclidean structure, matrices, linear transformations, invertible matrix, group structure, denotations (notational representations), topological structure, metric, open set (unit ball).

Definitions (Concepts): axioms of n-dimensional vector space, vector space, linear combinations, base, coordinates, linear transformation, general linear group of transformations, isomorphism, metric, open set (open ball), matrix series.

Propositions: (P1) The space of all $n \times n$ matrices, denoted as $M(n, R)$, is isomorphic to R^N, $N = n^2$. (P2) Defining properties of the metric. (P3) Relations between the Euclidean metric and topological structures. (P4) Criterion of the convergence of series in metric spaces.

Arguments: (A1.1) According to the proposition (P1) the vector space $M(n, R)$ is isomorphic to R^N, $N = n^2$, therefore the properties of this standard vector space permit the traditional assigning of coordinates to a vector and its Euclidean norm. Thus for any matrix element $A = (\boldsymbol{a_j^i})$, $i, j = 1,2, \dots n$, the Euclidean metric could be defined as $|A|^2 = \sum_{i,j} |a_j^i|^2$, where a_j^i are the coordinates of an element A in the standard basis E(ij). (A1.2) In order to define an open set, the interplay between the topological structure and Euclidean metric should be employed: since $M(n, R)$ is a linear space of square matrices $A = (\boldsymbol{a_j^i})$, a topological structure is determined by the Euclidean metric, which permits the definition of the unit ball in the space $M(n, R)$: $|X| < 1$, $X = (x_j^i)$.

Unitary practice 1 (UP1): this unitary practice contains the process of assigning coordinates with respect to the chosen base, introducing the Euclidean metric, verification of the axioms of metric defining open sets.

Arguments: (A2) Now the serious obstacles should be overcome if we would like to transfer the same structures to the subset of invertible matrices $GL(n, R)$ regarding its group structure: namely, how one should determine a neighborhood of unit element, and furthermore special attention should be paid to assure that all the matrices from a chosen neighborhood be still invertible. To guarantee that any matrix of a neighbourhood of unit element be invertible, the interplay with the higher algebra, using the division algorithm of unity I by (A - I), should be involved giving rise to matrix series which provides the inverse matrix. This algorithm suggests considering the product of matrices: $AB = (I + X)(I - X + X^2 - X^3 + \dots + (-1)^n X^n + \dots)$=I. This leads to introduce the inverse matrix in the form $A^{-1} = B$. Nevertheless, special caution should be taken to demonstrate that the series $I - X + X^2 - X^3 + \dots + (-1)^n X^n + \dots$ converge, in order to guarantee the inverse matrix be well defined. The question of converging of this matrix series plays a crucial role. A sequence of partial sums in the matrix space should be considered so that one could demonstrate that it is the fundamental sequence, i.e., the series converge. This will prove that the matrix $\mathrm{A} = (\mathrm{I} + \mathrm{X})$ is invertible when $|X| < 1$ and as result it is shown that, for $|X| < 1$, $A = (I + X) \in GL(n, R)$.

Unitary Practice (UP2). The corresponding unitary practice requires construction of the corresponding open unit ball in the neighbourhood of the unit matrix, in order to preserve the property of invertibility of matrices within such neighbourhood. The new coordinates $(\boldsymbol{x_j^i})$ of the matrix $A = (\boldsymbol{a_j^i})$, $i, j = 1,2, \dots n$, in the neighborhood $|A - E| < 1$ of unit element $E \in GL(n, R)$, are determined as $x_j^i(A) = a_j^i - \delta_j^i$, giving $x_j^i(E)$=0 as desired.

Arguments (A3). Naturally a question arises about the coordinates in a neighbourhood U_D centered at arbitrary element of $D \in GL(n, R)$. The group structure permits multiply every element of U_D by D^{-1}, so that the whole neighborhood U_D will be displaced to the neighborhood of the unit element, where the coordinates have been just determined.

Unitary Practice (UP3). The corresponding unitary practice involves interplay with the group structure of $GL(n, R)$ to construct the so-called local coordinate system. The real importance of the geometric nature will be disclosed in the next unitary practice.

3.2. Velocity of the curves in $GL(n, R)$

Problem-Situation 2.

Since $GL(n, R)$ is an open set in $M(n, R)$ in the topology introduced in previous considerations it is possible to define curves and try to determine the corresponding velocity vector, as in the traditional three-dimensional space.

Unitary practice (UP4) consists of description of the notions which are characteristic for R^N, such as continuity, differentials, curves, tangent vectors, superficies, in the new environment of $M(n, R)$ y $GL(n, R)$. A curve in $GL(n, R)$, may be considered as an element whose coordinates are differentiable functions of one parameter, say $A(t)$. We require $A(0) = E$, which means that the curve passes through the unit element when the parameter value is 0. As in the case of R^N, there should exist tangent vector (velocity), which in our new context is the derivative of the matrix $A(t)$, denoted as $A'(t)|_{t=0}$.

It is always a surprise for students that any matrix from $M(n, R)$ serves as the *tangent* to some element of $GL(n, R)$ (although they can visualize that any vector serves as velocity to some trajectory). The geometric meaning of this is that all vectors tangent to $GL(n, R)$ at unit element belong to the space of all matrices of order n.

4. Problems related to geometry

Naturally, similar questions there arise for the tangent spaces of $SL(n, R)$ and $O(n, R)$, where $O(n, R)$ and $SL(n, R)$ are subgroups of $GL(n, R)$ determined by specifications of the properties of its elements, dictated primarily by geometric considerations, and expressed through certain algebraic relations.

4.1. Geometry of the sets of matrices $SL(n, R)$ and $O(n, R)$

We give a brief description of exploration and geometric interpretations of the matrix groups $SL(n, R)$ and $O(n, R)$ and their tangent spaces**.**

Problem-Situation 3.

Describe the tangent spaces at unit element of the surfaces formed by the subsets $SL(n, R)$ and $O(n, R)$ in $GL(n, R)$.

Definitions: (D1) in the Euclidean space the system $f_i\ (u_1, u_2, \ldots, u_m), i = 1, \ldots, k$ describes a regular surface of dimension $d = m - r$ if the Jacobi matrix has rank r. (D2) The set $SL(n, R)$is determined by the requirement $\det\ A = 1$, i.e., there is only one equation (functional relation) which determines the surface of dimension $n^2 - 1$, which is called hypersurface, by definition (the rang of Jacobi matrix is equal to 1 here). (D3) The set $O(n, R)$ is determined by the requirement on its elements: $A^t\ A = E$.

Theorem: a surface is regular at a given point if its tangent space has dimension $d = m - r$, where the Jacobi matrix has a rank r.

Unitary practice (UP5) consists of counting dimensions of subspaces and finding tangent objects (using interplay with the continuity and differentiations). In particular, the set $O(2, R)$ of such matrices is characterized by three relations: they could be denoted $f_i(a, b, c, d) = 0,\ i = 1,2,3.$ And in the neighborhood of E three variables can be expressed in terms of the one variable: $a^2 + b^2 = 1,\ c^2 + d^2 = 1, (ac + bd) \quad = 0.$

The geometrical meaning is that this set represents a one dimensional subset (a curve) because the Jacobi matrix has rank 3.

The analogous consideration for $SL(n, R)$ gives a three dimensional surface in $M(n, R)$.

4.2. Construction of vector spaces tangent to the superficies of $SL(n, R)$ and $O(n, R)$ at unit element

The tangent vector to an orthogonal matrix A, $A^t A = I,$ is just a skew symmetric matrix. The dimensions coincide. In the case of unimodular matrices the tangent vectors are the matrices with the zero trace.

Both discoveries need corresponding neat treatment which bring a surprise to be able to differentiate a matrix and furthermore a determinant.

The unitary practice (UP6) establishes that $SL(n,R)$ is a regular superficies in the space of all matrices.

Procedure: First of all we demonstrate that the unit element (point E in $SL(n,R)$) is regular point of this hypersurface. It is sufficient to show that the tangent space for $SL(n,R)$ at this point has the dimension exactly n^2-1. Students have to consider an arbitrary curve $A(t) \in SL(n,R)$, passing through unit element when t is equal to zero: $A(0)=E$. Then this curve satisfy the polynomial relation $\det(A(t)) = 1$. Thus for the tangent element $X = \frac{d}{dt}(\det A(t))|_{t=0}$ we obtain the relation $Sp(X) = 0$. (Sp denotes the trace of matrix.) This result can be obtained by the direct differentiation of determinant taking into account that the coefficients of the equations are the elements of the Jacobi matrix $\partial\,(\det A)/\partial a_j^i$, $A = (\boldsymbol{a_j^i})$, when $A = E$. Thus the equation $Sp(X) = 0$ determines tangent space to $SL(n,R)$ at the unit point. It determines all the matrices with trace zero, and the dimension of this space is just n^2-1. Thus, it is proved that the unit point E is regular.

Students should demonstrate that any point B is regular (as in the (UP2).

4.3. Further algebraic constructions in the vector space of matrices

Problem-Situation 4 (applications to computer 3D Vision).

It easy to see that any 3d skew symmetric matrix is related to 3d-vector, and the vector product is related to Lie bracket $AB-BA$ in this case, thus our 3d vector space with the vector product is a Lie Algebra. The natural question arises, what is the corresponding local Lie group. This gives a simple way to introduce quaternions and obtain an algebraic operation on the sphere, which is norm preserving.

Such sort of questions are considered in optional courses of Differential Manifolds, Lie Groups and Lie Algebras serving as trivial examples, to which usually a reference is made without close considerations, leaving such tasks as individual work, because there are many special questions to treat within such courses.

5. Conclusions

The introduction of the unitary-systemic duality permits the reformulation of the “ingenious” vision that “there is the same mathematical object with the different representations” (as the matrices in our case) (Rondero & Font, 2015). Actually, there is just a complex system of practices which allow to solve the problems where the mathematical object “matrix” does not appear explicitly. There appear the representations of the matrices, different definitions of the matrices (not only as the simple arrangement of numbers), propositions and properties of them, procedures and technics which are applied to the specific sets of matrices and the corresponding argumentations. As a result of the process of the historical development there have been developed different epistemic configurations which involve the matrices so that some classes of them can be considered as a generalization of the initial concept.

Under this perspective, we consider the Theory of Ontosemiotic Approach as an adequate framework which allows us to analyse the learning process in the e-modality of Linear Algebra course in order to arrange the activities so that the students would achieve the deeper comprehension of the most important concepts and their mutual relations. We employ the notion of epistemic configurations in Onosemiotic Approach in order to realize an analysis of the mathematical practices performed by students in the process of problems solving. The series of activities which we have designed for our students require higher level of thinking, where the fusion of theoretical considerations, analytical methods and operative technics is required. These activities serve as a preliminary experience in research before their thesis projects.

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